# Global Uniform Asymptotic Stability of a Generalized Adaptive Bellman-Ford Algorithm

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Abstract—Self-stabilizing information spreading algorithms are a key basis block for building distributed system for device coordination. The adaptive Bellman-Ford (ABF) algorithm is a special case of these spreading algorithms. It finds the distance estimate of each node in a graph from a source set, but unlike the classical Bellman-Ford algorithm does not assume that all initial distance estimates exceed their true values. Though globally uniformly asymptotically stable (GUAS), its convergence can be very slow in graphs will short edges if some initial estimates are smaller than their true values. We propose here a generalization of ABF with additional parameters to permit faster convergence. We prove it to be GUAS, bounding the time to converge, and show via simulations that it withstands persistent bounded perturbations in the graph edge lengths.

## I. INTRODUCTION

In the last few decades the multi-agent systems literature has studied the stability of complex, networked, and distributed systems, [2]- [9]. Yet, this literature has yet to tackle stability issues associated with *open* systems like smart cities, the internet of things and tactical networks, all of which must support an unbounded and rapidly evolving collection of distributed services. They require stable, resilient, seamless and safe decentralized local coordination through low latency peer to peer communication, permitting devices to share and execute multiple applications and flexibly use local resources.

Their potential of open systems is limited by current constrained and inflexible device coordination (e.g., single-purpose IoT devices) and the frequent use of cloudlike remote infrastructure. Constraints on devices restrict reusability and the ability to be involved in multiple applications. Centralization engenders high latency, impairing the agility to exploit local infrastructure.

Aggregate computing is conceived to meet the challenge of open systems [12]. It uses a layered architecture [12] where the middle layer contains three classes of coordination blocks whose compositions, including feedback, realize many distributed services [10], [11]. These blocks are distributed graph algorithms and are: G-blocks that propagate information through a network; C-blocks that collect and summarize information to coordinating devices; and T-blocks that delay actions for stability.

Stability of their interconnections is critical to their proper functioning, but most studies have either been empirical [13], [15], or only demonstrated selfstabilization [17]. Unlike global uniform asymptotic stability (GUAS), [1], [19], [18], self-stabilization does not guarantee any robustness to even the smallest perturbations which are inevitable under feedback. The exceptions are [24], [25] and [26]. Among these [26] establishes the GUAS and the convergence time of a fairly, but not the most general, G-block. Under an additional Lipschitz condition it proves the ultimate boundedness of this Gblock under persistent perturbations in the edge lengths defining the neighborhoods in a network of devices.

On the other hand [24] and [25] consider a special case of the G-block studied in [26]– the Adaptive Bellman-Ford (ABF) algorithm—that adaptively finds the distance between all devices and a source set. Unlike the classical Bellman-Ford (CBF) algorithm [22], ABF does not assume that all initial distance estimates are overestimates, an assumption that precludes GUAS. Using a Lyapunov framework [24], [25] prove GUAS, with tight bounds on convergence time and ultimate boundedness under persistent perturbations. Ultimate bounds in the face of perturbations holds may allow the use of sophisticated refinements [23] of the small gain theorem [1], or for that matter the passivity theorem [20] or its offshoots, [21] to help establish stability under feedback.

As underestimates in ABF converge slowly, we formulate a special case of the most general G-block given in [17] which has additional parameters that can be tuned to achieve faster convergence. We call this algorithm the generalized ABF (GABF). Without a dead zone parameter  $D \ge 0$ , GABF performs poorly under structural perturbations, but D > 0 makes it robust to persistent perturbations at the price of slower convergence. Its analysis is nontrivial as unlike ABF no Lyapunov function has been found to prove GUAS. Yet we prove its GUAS, providing a conservative bound on its time to converge. Simulations confirm that GABF is robust to persistent perturbations when D > 0 and is still faster than ABF.

After motivating the Generalized ABF in Section II, we introduce the algorithm in Section III. Section IV has preliminaries, Section V has the proof of GUAS, Section VI the simulations and Section VII the conclusion. Proofs are omitted due to space constraints.

### II. BACKGROUND

The version of G-block that specializes to ABF was enunciated in [17] using field calculus language and proved as self-stabilizing for states in a noetherian ring. It was shown to be GUAS for real state in [26],

Suppose an undirected graph,  $\mathcal{G}$  has node set  $V = \{1, 2, \dots, N\}$ . An edge exists between nodes *i* and *k* if

they are *neighbors* and can communicate with each other. the set of neighbors of i is  $\mathcal{N}(i)$ . With  $x_i$  the desired state of the *i*-th node,  $\hat{x}_i(t)$  its *estimate* at time t, obeys

$$\hat{x}_i(t+1) = \min\left\{\min_{k\in\mathcal{N}(i)}\left\{f\left(\hat{x}_k(t), e_{ik}\right)\right\}, s_i\right\}, \forall t \ge t_0,$$
(1)

where  $e_{ik} > e_{\min} > 0$  is the edge length between *i* and *k* and  $s_i$  is the maximum value of  $\hat{x}_i(t)$ , possibly infinite.

The function  $f(\cdot, \cdot)$  is progressive i.e. for some  $\sigma > 0$ ,  $f(a, b) > a + \sigma$ , and obeys  $f(a_1, b) \ge f(a_2, b)$ , if  $a_1 \ge a_2$ . In [26] we proved that (1) is GUAS, i.e. the  $\hat{x}_i(t)$  converge to  $x_i$  if at least one  $s_i$  is finite, and that it is ultimately bounded under persistent perturbations in the  $e_{ik}$ .

A special case, ABF finds the distance of all nodes from a source set  $S \subset V$ . In particular the true distance  $d_i$  of node *i* from the source set *S* obeys,

$$d_i = \begin{cases} \min_{j \in \mathcal{N}(i)} \{ d_j + e_{ij} \} & i \notin S \\ 0 & i \in S \end{cases}$$
(2)

where  $e_{ij}$  are the edge lengths. Mimicking these relations, ABF adaptively updates the estimate  $\hat{d}_k(t)$  of  $d_k$ , using

$$\hat{d}_i(t+1) = \begin{cases} \min_{j \in \mathcal{N}(i)} \left\{ \hat{d}_j(t) + e_{ij} \right\} & i \notin S \\ 0 & i \in S \end{cases}, \quad (3)$$

 $\forall t \geq t_0$ . Thus f(a, b) = a + b,  $\hat{d}_i = \hat{x}_i$ ,  $s_i = 0$  if  $i \in S$  and  $s_i = \infty$  for all  $i \notin S$ . Unlike CBF, ABF permits  $\hat{d}_i(t_0)$  to be smaller than its true value  $d_i$ , and multiple sources. We have analyzed ABF in a Lyapunov framework in [24] and [25] and given tight ultimate error bounds under perturbations and tight bounds on the time to attain them.

Define the distance estimation errors,

$$\Delta_i(t) = \hat{d}_i(t) - d_i, \tag{4}$$

the least underestimate  $\Delta^{-}(t) = \max[0, -\min_i \Delta_i(t)]$ and  $\Delta^{+}(t) = \max[0, \max_i \Delta_i(t)]$ , the greatest overestimate. Then [24] shows that in ABF both  $\Delta^{-}(t)$  and  $\Delta^{+}(t)$  are nonincreasing and that there is a T such that they each decrease by a minimum amount (unless they become zero) at least every T iterations. Thus,  $L(t) = \Delta^{+}(t) + \Delta^{-}(t)$  is a Lyapunov function for ABF.

We show in [25], that while  $\Delta^+(t)$  falls rapidly to zero, the convergence time for  $\Delta^-(t)$  may be slow, being upper bounded by  $\lceil (d_{\max} - \hat{d}_{\min}(t_0))/e_{\min} \rceil$  where  $d_{\max}$  and  $\hat{d}_{\min}(t_0)$  are the largest distance in the graph and the smallest initial estimate respectively. Thus if  $e_{\min}$  is the edge length between two nodes whose initial distance estimates are underestimates, then their estimation errors rise by  $e_{\min}$  at a time. Convergence is slow when  $e_{\min}$ is small. GABF overcomse this difficulty, by providing additional parameters that increase the convergence rate.

### III. GENERALIZED ABF

The most general G-block in [17] obeys

$$\hat{x}_i(t+1) = F\left(\min\left\{\min_{k\in\mathcal{N}(i)}\left\{f\left(\hat{x}_k(t), e_{ik}\right)\right\}, s_i\right\}, \hat{x}_i(t), v_i\right)\right)$$
(5)

where  $v_i$  are certain environmental variables. The function  $F(\ell_1, \ell_2, v)$  obeys for some M and  $\delta > 0$ ,

$$F(\ell_1, \ell_2, v) \begin{cases} = \ell_1 & \ell_1 = \ell_2 \text{ or } \ell_2 > M \\ \ge \ell_2 + \delta & \text{otherwise} \end{cases}$$
(6)

Thus in (1)  $M = -\infty$ . In (5) and (6),  $\hat{x}_i(t) = \ell_2$ . The second bullet in (6) permits a faster initial ascent of  $\hat{x}_i(t)$ .

We now enunciate a special case of (5) where  $d_i(t)$  the current estimate of the distance  $d_i$  of node *i* from the source set *S*, evolves for all  $t \ge t_0$ , as

$$\tilde{d}_{i}(t+1) = \begin{cases} \min_{j \in \mathcal{N}(i)} \{ \hat{d}_{j}(t) + e_{ij} \}, & i \notin S \\ 0, & i \in S \end{cases}.$$
(7)

Then distance estimates  $\hat{d}_i(t)$  evolves as:

$$\hat{d}_{i}(t+1) = \begin{cases} \tilde{d}_{i}(t+1), & \tilde{d}_{i}(t+1) = \hat{d}_{i}(t) \text{ or } \hat{d}_{i}(t) > M \\ g(\hat{d}_{i}(t)), & \text{otherwise} \end{cases}$$
(8)

where M is a finite number and the strictly increasing g(x) is finite finite x and obeys for some  $\delta > 0$ ,

$$g(x) \ge x + \delta. \tag{9}$$

Thus unlike the ABF, (7-9) does not assume that distance estimates of the sources are anchored to zero. Further,

$$\tilde{d}_i(t_0) \ge 0 \text{ and } \hat{d}_i(t_0) \ge 0, \ \forall i \in V.$$
 (10)

Initially, most estimates obey the second bullet of (8) and for large  $\delta$  rise rapidly until exceeding M, when the first bullet is invoked. While the second may again be invoked, we show that there comes a time after which using the first bullet yields immediate convergence.

Our proof also shows that underestimates are eventually eliminated. Though not explicitly quantified in our proof, in generic networks and large M and  $\delta$  this elimination is rapid, especially if M exceeds the graph diameter. In such cases the rising value problem that slows the convergence of underestimates in ABF is obviated. As our proof is for all M > 0 and  $\delta > 0$ , it provides a conservative estimate of the convergence time. However, simulations confirm fast convergence for large M and  $\delta$ .

Yet we argue that (8) and indeed (5) are fundamentally nonrobust to perturbations in the edge lengths. This is so as when  $M > \hat{d}_i$ , update using the first bullet requires the precise satisfaction of  $\tilde{d}_i(t+1) = \hat{d}_i(t)$ . Because of (7) this cannot be sustained under perturbations in  $e_{ij}$ , resulting in the repeated use of the second bullet and the rise to M, making M the ultimate error bound. Thus instead of (8) we use for some dead zone  $D \ge 0$ 

$$\hat{d}_i(t+1) = \begin{cases} \tilde{d}_i(t+1), \ |\tilde{d}_i(t+1) - \hat{d}_i(t)| \le D \\ \text{or } \hat{d}_i(t) > M \\ g(\hat{d}_i(t)), \text{ otherwise} \end{cases}$$
(11)

Indeed (7,9,11) is GABF. Of course (8) is a special case of ' (11), with D = 0. Simulations confirm that GABF is ultimately bounded with small bounds under perturbations

in  $e_{jk}$  that are smaller than D in magnitude. A larger D tolerates larger perturbations, but brings GABF closer to ABF slowing convergence. This is expected. Faster systems are high pass filters that amplify effects of noise.

The convergence analysis of this algorithm is nontrivial as unlike ABF  $\Delta^+(t) + \Delta^-(t)$  can increase, and is thus not a Lyapunov function. The proof of GUAS in the next section does not use a Lyapunov function at all.

There is a subtle difference between the assumptions made in [17], and those made here. In [17],  $\ell_i$  are assumed to lie in a Noetherian ring with maximal element M. Translated to GABF this means that one *a priori* assumes the boundedness of the states of the algorithm with M serving as an upper bound. This assumes that one knows the largest possible distance in the network and that M exceeds that value. As open networks can grow unpredictably this is an unappealing assumption.

In contrast we do not assume that the distance estimates are a priori bounded. Nor do we assume that Mexceeds the largest distance estimate. Thus the proof of self-stabilization given in [17] does not apply to GABF even when D = 0. Our proof of GUAS only assumes that  $M > 0, D \ge 0, \delta > 0$ , the graph is connected and the source set S is nonempty.

As all edges are positive (10) is entirely reasonable. With some abuse of terminology global stability will involve convergence for all initial conditions obeying (10).

## IV. PRELIMINARIES

Assumption 1 is the standing assumption of this paper. Assumption 1: Graph  $\mathcal{G}$  is connected,  $S \neq \emptyset$ ,  $S \neq V$ ,  $e_{\max} \ge e_{ik} = e_{ki} \ge e_{\min} > 0$ ,  $D \ge 0$  and  $g(\cdot)$  obeys the conditions given below (11).

As in any iteration the estimated distance of a node is obtained by one of the bullets in (11), at each t, we partition V into two sets defined below.

Definition 1: The ABF set  $\mathcal{A}(t)$  comprises all nodes that use the first case in (11) to obtain  $\hat{d}_i(t+1)$ . Define a set the set of extraordinary nodes  $\mathcal{E}(t) = V \setminus \mathcal{A}(t)$  to be those that use the second case in (11) to obtain  $\hat{d}_i(t+1)$ .

We also introduce the notion of a constraining node of i that determines  $\hat{d}_i(t+1)$ .

Definition 2: For  $i \in \mathcal{A}(t)$ , the minimizing j in (7) used to find  $\tilde{d}_i(j+1)$ , is *i*'s current constraining node at t. For  $i \in \mathcal{E}(t)$ , i is its own current constraining node. Sources are their own constraining nodes.

It is important to note that a source may well be in the extraordinary set  $\mathcal{E}$ . The next definition introduces a *true constraining node*.

Definition 3: A j that minimizes the right hand side of (2) is a true constraining node of  $i \in V \setminus S$ . As a node may have multiple true constraining nodes, the set of true constraining nodes of a node  $i \in V \setminus S$  is C(i).

Based on Definition 3, we introduce another.

Definition 4: Call a path from a node i to the source set a shortest path, if it starts at i, ends with a source node and each node in the path is a true constraining node of its predecessor. Call a shortest path from i, the longest shortest path if it has the most nodes among all shortest paths of i. The set  $\mathcal{F}_i$  is the set of nodes whose longest shortest paths to the source have i+1 nodes. Call  $\mathcal{D}(\mathcal{G})$  the effective diameter of  $\mathcal{G}$  if the longest shortest path among all  $i \in V$  has  $\mathcal{D}(\mathcal{G})$  nodes.

Thus if a node *i* has two shortest paths one having two and the other three nodes then  $i \notin \mathcal{F}_1$  but  $i \in \mathcal{F}_2$ . From Definition 4,  $\mathcal{F}_i$  follows

$$\mathcal{F}_0 = S. \ \mathcal{F}_i \neq \emptyset, \ i \in \{0, 1, ..., \mathcal{D}(\mathcal{G}) - 1\}.$$
(12)

Every node in  $\mathcal{F}_{i+1}$  has a true constraining node in  $\mathcal{F}_i$ :

$$\mathcal{C}(j) \bigcap \mathcal{F}_i \neq \emptyset, \ \forall \ j \in \mathcal{F}_{i+1}.$$
(13)

Further it has been shown in [25] that  $\mathcal{D}(\mathcal{G})$  is finite.

# V. GLOBAL UNIFORM ASYMPTOTIC STABILITY

The first lemma asserts that distances estimates of all nodes are upper bounded.

Lemma 1: Consider (7), (9) and (11) under Assumption 1 and all initialization as in (10). Then for  $\forall i \in \mathcal{F}_j$  defined in Definition 4, and  $e_{\max}$  in Assumption 1,

 $\hat{d}_i(t) \le \max\{d_i, g(g(M))\} + je_{\max}, \ \forall t \ge j+1.$  (14) We define two sets requiring recursive definitions.

Definition 5: The set of nodes  $\mathcal{R}(t)$  rooted to S obeys  $\mathcal{R}(t_0) = S$ . Further  $\mathcal{R}(t+1)$  comprises all nodes whose current constraining node at t+1, is in  $\mathcal{R}(t)$ . We also define a set  $\mathcal{U}(t)$  as unrooted to the source set if  $\mathcal{U}(t_0) = V \setminus S$ , and  $\mathcal{U}(t+1)$  comprises all nodes whose current constraining node at t+1, is in  $\mathcal{U}(t)$ .

Then lemma 2 shows that  $\mathcal{U}(t)$  and  $\mathcal{R}(t)$ , partition V.

Lemma 2: Consider (7), (9) and (11) under Assumption 1, with  $\mathcal{R}(t)$  and  $\mathcal{U}(t)$  defined in Definition 5. Then for all initialization as in (10) and for all t,

$$\mathcal{U}(t) \bigcup \mathcal{R}(t) = V$$
, and  $\mathcal{U}(t) \bigcap \mathcal{R}(t) = \emptyset$ . (15)  
Observe from the definition of the unrooted set that

$$\mathcal{U}(t) = \emptyset \Rightarrow \mathcal{U}(t+1) = \emptyset.$$
(16)

This is so because at t + 1 every node *i* has a current constraining node *j*. As  $\mathcal{U}(t)$  is empty, from Lemma 2,  $j \in \mathcal{R}(t)$  and thus  $i \in \mathcal{R}(t+1)$ . We further define the following function:

$$\hat{d}_{\min}(t) = \min_{j \in \mathcal{U}(t)} \{ \hat{d}_j(t) \} \text{ if } \mathcal{U}(t) \neq \emptyset.$$
(17)

The next lemma asserts that the lower bound of  $\hat{d}_i(t)$  with  $i \in \mathcal{U}(t)$  strictly increases.

Lemma 3: Consider (7), (9) and (11) under Assumption 1,  $\mathcal{U}(t)$  and  $\hat{d}_{\min}(t)$  defined in Definition 5 and (17), respectively, for all initialization as in (10). The following holds while the set  $\mathcal{U}(t) \neq \emptyset$ :

 $\hat{d}_i(t) \ge \hat{d}_{\min}(t_0) + \min\{e_{\min}, \delta\}(t - t_0), \ \forall i \in \mathcal{U}(t).$  (18)

A direct consequence of these lemmas is the fact that all nodes are eventually rooted to the source set.

We now assert that no estimates of nodes in  $\mathcal{R}(t)$  are underestimates.

Lemma 4: Consider (7), (9) and (11) under Assumption 1, with  $\mathcal{A}(t)$ ,  $\mathcal{E}(t)$ ,  $\mathcal{R}(t)$  and  $d_i$  defined in Definition 1, 5 and (2), respectively. Then for all initialization as in (10) and  $i \in \mathcal{R}(t)$ ,  $\hat{d}_i(t)$  obeys

$$\hat{d}_i(t) \ge d_i \tag{19}$$

Define:

$$d_{\max} = \max_{k \in V} \{d_k\} \tag{20}$$

and

$$T^* = \left\lceil \frac{d_{\max}}{\min\{\delta, e_{\min}\}} \right\rceil.$$
(21)

Then from Lemma 3, we have that

$$d_i(t) \ge d_i, \ \forall i \in \mathcal{U}(t), \ \forall t \ge t_0 + T^*.$$

Thus, as  $\mathcal{U}(\cdot)$  and  $\mathcal{R}(\cdot)$  partition V, from Lemma 4,

$$\hat{d}_i(t) \ge d_i, \ \forall i \in V, \ \forall t \ge t_0 + T^*.$$
(22)

Thus underestimates are eliminated at  $T^*$ . We now define the smallest distance in  $\mathcal{F}_i$  as

$$d_{i\min} = \min_{j \in \mathcal{F}_i} \{d_j\}.$$
 (23)

Observe  $d_{0\min} = 0$ , as  $\mathcal{F}_0 = S$ . Define a sequence

$$T_i = \max\left\{0, \left\lceil \frac{M - d_{i\min} - D}{\delta} \right\rceil\right\} + 1.$$
 (24)

Then we have the following Lemma.

Lemma 5: Consider (7), (9) and (11) under Assumption 1, and  $\mathcal{F}_i$  defined in Assumption 4. Suppose at a time  $t_L > T^*$ , defined in (22) and  $L \in \{0, 1, \dots, \mathcal{D}(\mathcal{G}) - 2\}$ 

$$\hat{d}_i(t) = d_i, \ \forall i \in \bigcup_{i=0}^L \mathcal{F}_i, \forall t \ge t_L,$$
(25)

i.e. all distance estimates of nodes in  $\mathcal{F}_0, \dots, \mathcal{F}_L$  have converged by  $t_L$ . Then with  $T_i$  defined in (24), there holds:

$$\hat{d}_i(t) = d_i, \ \forall i \in \bigcup_{i=0}^{L+1} \mathcal{F}_i, \forall t \ge t_L + T_{L+1}.$$
(26)

Then the main theorem below proves GUAS and furnishes an upper bound on the convergence time.

Theorem 1: Consider (7), (9) and (11) under Assumption 1, with  $T_i$  defined in (24) and  $T^*$  in (21) define

$$T^* = \max\{T^*, T_0 - 1\}.$$
 (27)

Then for all initialization as in (10), for all  $i \in V$ ,

$$\hat{d}_i(t) = d_i, \ \forall t > t_0 + T^* + \sum_{i=1}^{\mathcal{D}(G)-1} T_i.$$
 (28)

Several comments are in order. Convergence is uniform as  $T^*$  and  $T_i$  are independent of the initial time  $t_0$ . The time to converge is very conservative. In particular  $T^*$  can be large for small  $e_{\min}$  which appears in  $T^*$  as our analysis allows that even if  $\delta$  and or M is large, underestimates are persistently in  $\mathcal{E}(\cdot)$ , i.e. the first bullet of (11) is invoked repeatedly. In practice, unless D is large, large M and  $\delta$  induce the second bullet to be invoked far more often than the first. In such a case, especially with  $M = \delta > d_{\max}$ ,  $T^* = 1$  and  $T_i = 2$  and convergence occurs in an effective diameter time.

Yet as in all dynamical systems, this must be balanced against response to perturbation. Systems with fast dynamics tend not to smooth perturbations. Indeed as explained in Section III, with D = 0, any persistent perturbation in the edges will cause estimates to persistently rise to M. Simulations confirm that this is prevented if perturbations are less than D in magnitude, improving GABF's response to noise. Large D on the other hand makes GABF closer to ABF and increases the time to converge. This trade off between convergence rate and perturbation response pervades most algorithms. To summarize for an unperturbed graph choose a large  $M = \delta$ and D = 0. If you expect perturbations, raise D to exceed the perturbation bound.

# VI. SIMULATIONS

We now compare the relative performance of ABF and GABF. In the simulations 500 nodes, one of which is a source, are randomly distributed in a  $4 \times 1 \text{ km}^2$ field, communicating over a 0.25 km radius, running synchronously. The initial distance estimates are picked as  $\hat{d}_i(0) \in U(0, \sqrt{17})$ km. Each simulation is run 10 times.

In the sequel  $\Delta_i, \Delta^-$  and  $\Delta^+$  are as in (4) and below (4). Thus,  $\Delta^+(t) = \Delta^-(t) = 0$  indicates convergence at time t. The simulations show that unlike ABF,  $\Delta^-$  and  $\Delta^+$  need not be non-increasing in GABF.

# A. Comparison between GABF and ABF

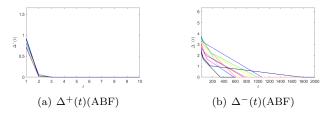


Fig. 1. Convergence time in ABF for (a) the greatest overestimate  $\Delta^+(t)$  and (b) the least underestimate  $\Delta^-(t)$  for 10 runs without perturbations.

Figure 1 depicts the performance of ABF: While  $\Delta^+(t)$  converges within  $\mathcal{D}(\mathcal{G})$  rounds,  $\Delta^-(t)$  constrained by the "rising value problem" takes much longer to converge. Figure 2 concerns GABF with  $M = \delta = 2\sqrt{17}$ km, D = 0, i.e. with  $M > d_{\text{max}}$ :  $\Delta^-(t)$  is rid of the "rising value problem" and converges in 2 rounds;  $\Delta^+(t)$ , takes slightly longer than ABF. Yet the overall convergence is much faster than ABF.

In Figure 3 with D = 0.06, GABF converges slower than with D = 0. However, compared with ABF in Figure 1, the convergence of GABF is still much faster.

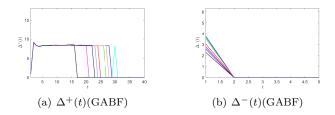


Fig. 2. With  $M = \delta = 2\sqrt{17}$ , D = 0, convergence time for (a) the greatest overestimate  $\Delta^+(t)$  and (b) the least underestimate  $\Delta^-(t)$  for 10 runs without perturbations.

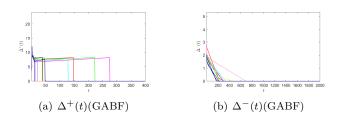


Fig. 3. With  $M = \delta = 2\sqrt{17}$ , D = 0.06, convergence time for (a) the greatest overestimate  $\Delta^+(t)$  and (b) the least underestimate  $\Delta^-(t)$  for 10 runs without perturbations.

## B. Effect of M and $\delta$

We first investigate the effect of a small M. In this case, we choose  $M = \delta = 0.1$ km and D = 0, thus M is smaller than true distances of most nodes.

Figure 4 shows that with  $M = \delta = 0.1$ km, i.e. both smaller than the  $d_i$ , and D = 0, the behavior of  $\Delta^+(t)$ and  $\Delta^-(t)$  are very close to that in ABF. Evidently with small M, most nodes use the first bullet of (11) and the rising value problem persists. Also the use of the second bullet in (11), may cause  $\Delta^+(t)$  not to converge in  $\mathcal{D}(\mathcal{G})$ rounds. Figure 5 shows the simulation results with M = $\delta = 2$ km, D = 0. In this case, M is slightly below  $d_{\text{max}}$ , and GABF converges faster.

Figure 6 involves  $M = 2\sqrt{17}$ km,  $\delta = 0.1$ km and D = 0. In this case,  $\Delta^{-}(t)$  still converges faster than in ABF. However  $\Delta^{+}(t)$  needs a much longer time than with a larger  $\delta$  as estimates take longer to rise to M.

## C. Comparison under perturbations

Consider static nodes with asymmetric noise in the estimated  $e_{ij}$ , then edge lengths used in (7) effectively

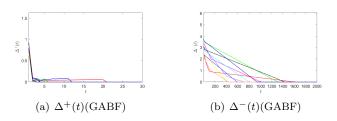


Fig. 4. With  $M = \delta = 0.1, D = 0$ , convergence time for (a) the greatest overestimate  $\Delta^+(t)$  and (b) the least underestimate  $\Delta^-(t)$  for 10 runs without perturbations.

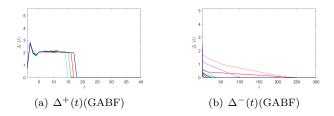


Fig. 5. With  $M = \delta = 2, D = 0$ , convergence time for (a) the greatest overestimate  $\Delta^+(t)$  and (b) the least underestimate  $\Delta^-(t)$  for 10 runs without perturbations.

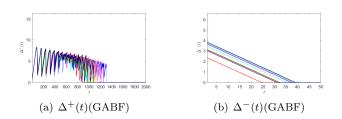


Fig. 6. With  $M = 2\sqrt{17}$ ,  $\delta = 0.1$ , D = 0, convergence time for (a) the greatest overestimate  $\Delta^+(t)$  and (b) the least underestimate  $\Delta^-(t)$  for 10 runs without perturbations.

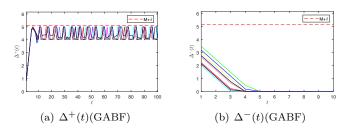


Fig. 7. With  $M = \sqrt{17}, \delta = 1, D = 0$ , comparison between (a) the greatest overestimate  $\Delta^+(t)$  and  $M + \delta$ , and (b) comparison between the least underestimate  $\Delta^-(t)$  and  $M + \delta$  for 10 runs under perturbations.  $\epsilon = e_{\min}$ , and  $e_{\min}$  is ranging from  $1.2 \times 10^{-4}$  to  $4.0 \times 10^{-3}$  for 10 trials.

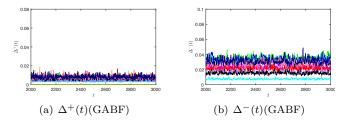


Fig. 8. With  $M = \delta = \sqrt{17}$ , D = 0.06, (a) the greatest overestimate  $\Delta^+(t)$  and (b) the least underestimate  $\Delta^-(t)$  for 10 runs under perturbations.  $\epsilon = e_{\min}$ , and  $e_{\min}$  is ranging from  $1.2 \times 10^{-4}$  to  $4.0 \times 10^{-3}$  for 10 trials.

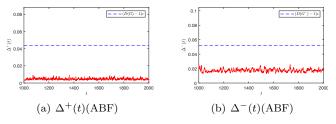


Fig. 9. Comparison between (a) the greatest overestimate  $\Delta^+(t)$ and  $(\mathcal{D}(\mathcal{G}) - 1)\epsilon$ , and (b) between the least underestimate  $\Delta^-(t)$ and  $(\mathcal{D}(\mathcal{G}^-) - 1)\epsilon$  under perturbations.  $\epsilon = e_{\min} = 3.7 \times 10^{-3}$ .

change from their nominal value  $e_{ij}$  as  $\bar{e}_{ij}(t) = e_{ij} + \epsilon_{ij}(t)$ . For some  $\epsilon$ 

$$|\epsilon_{ij}(t)| < \epsilon < e_{\min},\tag{29}$$

ensuring no edge length is negative. The noise is asymmetric i.e.  $e_{ij}(t) \neq e_{ji}(t)$ . Without asymmetry this also models persistent perturbations in  $e_{ij}$ .

In all simulations below,  $\epsilon_{ij}(t) \in \mathcal{U}(-\epsilon, \epsilon)$ , with  $\epsilon = e_{\min}$ , and  $e_{\min} = 1.2 \times 10^{-4}, 3.7 \times 10^{-3}, 1.3 \times 10^{-3}, 3.4 \times 10^{-3}, 3.7 \times 10^{-3}, 3.7 \times 10^{-3}, 2.1 \times 10^{-3}, 2.6 \times 10^{-3}, 3.9 \times 10^{-3}$  and  $4.0 \times 10^{-3}$  for the 10 trials.

Figure 7 is for GABF with  $M = \sqrt{17}$ km >  $d_{\text{max}}$ , D = 0and  $\delta = 1$ . In this case,  $\Delta^+(t)$  is orders of magnitude bigger than the perturbations. Predictably both  $\Delta^+$  and  $\Delta^-$  are upper bounded by  $M+\delta$ . Figure 7 (a) particularly confirms the persistent use of the second bullet in (11).

In Figure 8, D = 0.06:  $\Delta^+(t)$  and  $\Delta^-(t)$  are both ultimately upper bounded by  $(\mathcal{D}(\mathcal{G}) - 1)\epsilon$  and  $(\mathcal{D}(\mathcal{G}^-) - 1)\epsilon$ , respectively, with  $\mathcal{G}^-$  the graph obtained by reducing the edges to the smallest values. These are also the *tight* ultimate bounds for ABF given in [25]. Though the actual bounds for ABF are smaller (see Figure 9), estimates given by GABF with D = .06 still falls below the ultimate bounds for ABF derived in [25].

## VII. CONCLUSION

We have enunciated a new generalization of the ABF (GABF) with three parameters M,  $\delta$  and D. For D = 0 it is a special case of the most general G-block in [17]. For suitably chosen M,  $\delta$  and D, GABF removes a weakness of ABF: the slow rise of underestimates in face of small edge lengths. Unlike [17], we do not assume that the states of GABF are *a priori* bounded. Nor do we assume that the parameter M exceeds the largest distance in the network. Instead, we prove that the algorithm is GUAS as long as  $\delta > 0$ , M > 0 and  $D \ge 0$ , the graph is connected, and the source set is nonempty.

We argue and numerically confirm that with D = 0, GABF is not robust to bounded perturbations, and that robustness obtains if D exceeds the bound on these perturbations. This comes at the expense of slower convergence that is still faster than ABF. Deriving ultimate bounds under persistent perturbations as done for ABF in [25], is an area of future work.

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