A Lyapunov Analysis of a Most Probable Path Finding Algorithm

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***Abstract*—Distributed information spreading algorithms are important building blocks in Aggregate Computing. We consider a special case, namely for finding a most probable path for message delivery from a set of sources to each device in a network. We formulate a Lyapunov function to prove its regional stability subject to initialization of estimated probabilities to the natural interval $[0, 1)$. We also prove that the algorithm converges in a finite time, and is ultimately bounded under persistent measurement errors. We provide tight bounds for convergence time, the ultimate bound, and the time for its attainment.**

***Index Terms*—Lyapunov function, regional stability, aggregate computing, ultimate bounds.**

I. Introduction

AGGREGATE Computing [1] has been proposed for device coordination in open complex networks like smart cities, edge computing systems, and the Internet of Things (IoT) [2]. Such systems involve compositions of distributed algorithms. Being open, they must accommodate an unbounded number of devices and services. Each device shares multiple tasks with neighbors by seamlessly and safely communicating locally, often under feedback, and is subject to perturbations.

As ordinary distributed algorithms focus overly on message construction and passing, the analyses of their compositions are typically limited to self-stabilization. None exist for transient performance like time to converge or bounds under persistent perturbations, needed to guarantee safe operations. *Aggregate Computing* [1] avoids these problems by making *message passing details implicit and enforcing strict abstraction barriers information access*. This allows each component of a distributed system to be analyzed separately.

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Compositions of three basis blocks realize most coordination tasks: G block spreads information from sources across a network, C collects information from devices to sources, and T performs timing operations. Each is a self-stabilizing distributed algorithm [1].

Recently we have analyzed G blocks beyond self-stabilization; e.g., [3] considers a fairly general G block and proves its uniform asymptotic stability, providing a bound on the convergence time, proving it to be ultimately bounded under perturbations, quantifying the ultimate bounds and a bound on the time to attain them. *None of the bounds provided in [3] are tight.* Nor is a Lyapunov function used. As explained in [4], providing tight bounds is important, as is the use of a Lyapunov function, which can permit closed loop analysis using variants of the small gain theorem, [5].

Special cases often permit tighter results. Indeed, [4] considers the *Adaptive Bellman-Ford (ABF) algorithm*, a special G block, which finds the distance of each device in a network from a source set. For ABF, [4] provides a Lyapunov function, proves ultimate boundedness under persistent perturbations, and provides tight bounds on the time to converge, the ultimate bounds, and the time to attain them.

We show that the Lyapunov function of [4] also applies to another special case for finding a *Most Probable Path (MPP)* between a set of sources and each device in a network. Given the probabilities of correct transmission through *independent* edges in a network of devices, this finds the smallest probabilities of failed transmission from a source set to each device in a network and, *implicitly*, a corresponding path.

Like the shortest path problem [6], [7], MPP has been used for routing in cognitive radio and [8], network-driven contagion phenomena [9], DNA sequencing [10], language processing [11], and temporal networks [12]. These either *require initial estimates to be overestimates, unsustainable under perturbations*, or cannot provide convergence time or prove robustness. While one can find the largest probability of transmission using ABF applied to negative logarithms of probabilities, the results on tight ultimate bounds derived in [4] do not hold. Tight bounds under additive perturbations in the edge probabilities are critical to setting up Lyapunov based small gain variants as in [5] toward our eventual goal of analyzing G blocks under feedback.

With the Lyapunov function of [4], we prove here that the MPP Algorithm (MPPA) is *regionally stable*: Convergence is assured if the initial probability estimates are in $[0, 1)$. We also prove that MPPA is ultimately bounded under persistent perturbations in the edge probabilities, e.g., due to measurement/communication noise or feedback and bound the time to attain these bounds. Bounds on the convergence time, the ultimate bounds and the time to attain them are *tight.*

In the remainder of this letter, Section II presents the algorithm, Section III the Lyapunov function, Section IV the convergence time, Section V the robustness analysis, Section VI provides simulations, and Section VII concludes.

II. Preliminaries and the Problem

We model the network of devices as an undirected graph $G = (V, E)$, with $V = \{1, 2, \ldots, N\}$ the set of nodes or devices, and E the set of edges, i.e., communication links between nodes. Node j is a neighbor of node i if there is an edge between i and j. We define $\mathcal{N}(i)$ as the set of neighbors of node i, and e_{ij} *is the probability of successful transmission between* neighbors i and j. The e_{ij} are independent and obey

$$0 < e_{\min} \leq e_{ij} \leq e_{\max} < 1, \;\; \forall i \in V \text{ and } j \in \mathcal{N}(i). \qquad (1)$$

Let $S \subsetneq V$ be the source set. There may be multiple paths from a source to a node with the same failure probability, e.g., if a source has two-edge paths to a node with each edge a probability of 0.5. *Thus, the most probable path is not unique*. The minimum probability of failed transmission (henceforth called *probability of failure*) from S to a node, designated as p_i *is however unique. Thus we will find the smallest probability of failure. One of the MPPs will emerge as a byproduct.*

Under independence, the probability of failure at i in transmitting through j is $1 - (1 - p_j)e_{ij}$. Thus from Bellman's principle of optimality [7], assuming probability of failure from S to itself is zero, p_i *the true smallest probability of failure from* S to i obeys:

$$p_i = \begin{cases} \min\limits_{j\in\mathcal{N}(i)} \{1 - (1 - p_j)e_{ij}\} & i \notin S \\ 0 & i \in S. \end{cases} \qquad (2)$$

Definition 1: A minimizing j in the first bullet of (2) used to find p_i, is i's true constraining node. The set of true constraining nodes of a node $i \in V \setminus S$ is defined as $\mathcal{C}(i)$. A source is its own true constraining node.

Then an MPP to i is any sequence of true constraining nodes from S to i. Define $\hat{p}_i(t)$ as the estimate of p_i. On the basis of these estimates the minimum error probability from S to i at $t + 1$ is

$$\hat{p}_i(t+1) = \begin{cases} \min\limits_{j\in\mathcal{N}(i)} \{1 - (1 - \hat{p}_j(t))e_{ij}\} & i \notin S \\ 0 & i \in S. \end{cases} \qquad (3)$$

This defines MPPA. Consider,

Definition 2: Call $c_i(t+1) = \arg\min_{j\in\mathcal{N}(i)}\{1 - (1 - \hat{p}_j(t))e_{ij}\}$, i's constraining node at $t + 1$.

A sequence of $c_k(t+1)$ *starting from S to* ℓ *is thus an estimated MPP to* ℓ *at a given iteration.* We show later that each $\hat{p}_i(t)$ converges to the unique p_i and thus $c_i(t)$ will converge to a true constraining node of i. Even when MPPs to i start from multiple sources one of these will be chosen depending on initialization and the chosen $c_i(t)$.

Assumption 1: The graph $G = (V, E)$ is connected, undirected, $S \neq V$, with e_{ij} obeying (1). Further at the initial time t_0, $\forall i \in S, \hat{p}_i(t_0) = 0$ and $\forall i \in V \setminus S, \hat{p}_i(t_0) \in [0, 1)$.

A simple induction on time t can prove that, with the initial estimates defined in Assumption 1, $\hat{p}_i(t)$ under (3) obeys $0 \leq \hat{p}_i(t) < 1$ for all $t \geq t_0$ and $i \in V$.

The following definition is crucial:

Definition 3: Consider any sequence of nodes in G with each a true constraining node of its successor. The effective diameter of G, $\mathcal{D}(G)$, is the length of longest such sequence.

The proof showing that $\mathcal{D}(G)$ is finite is similar to [4, Lemma 1] and is thus omitted. Further, p_i in (2) obeys $0 \leq p_i < 1$ for all $i \in V$. This proof is also omitted.

III. A Lyapunov Function

We now present a Lyapunov function. Define the estimation error at node i as $\Delta_i(t) = \hat{p}_i(t) - p_i$. Further, define the greatest overestimate of the error $\Delta^+(t)$ and the least underestimate of the error $\Delta^-(t)$ below,

$$\Delta^+(t) = \max\left[0, \max_i \Delta_i(t)\right],$$
$$\Delta^-(t) = \max\left[0, -\min_i \Delta_i(t)\right]. \qquad (4)$$

Their sum forms our Lyapunov function:

$$L(t) = \Delta^+(t) + \Delta^-(t). \qquad (5)$$

Clearly, $L(t) \geq 0$ as $\Delta^+(t) \geq 0$ and $\Delta^-(t) \geq 0$, and equality holds iff for all $i \in V$, $\Delta_i(t) = 0$. Define

$$\mathcal{K}_+(t) = \left\{i \in V | \Delta_i(t) = \Delta^+(t)\right\} \text{ and} \qquad (6)$$
$$\mathcal{K}_-(t) = \left\{i \in V | \Delta_i(t) = -\Delta^-(t)\right\}. \qquad (7)$$

Lemma 1 shows that a nonzero $\Delta^+(t)$ must decrease.

Lemma 1: Consider (3) under Assumption 1. With Δ^+ defined in (4), if $\Delta^+(t) \neq 0$, then for all t,

$$\Delta^+(t+1) \leq e_{\max}\Delta^+(t). \qquad (8)$$

Proof: Consider $l \in \mathcal{K}_+(t+1)$. Suppose $j \in \mathcal{N}(l)$ is a true constraining node of l, i.e., from (2) and Definition 1,

$$p_l = 1 - (1 - p_j)e_{lj}. \qquad (9)$$

Then from (6),

$$\Delta^+(t+1) \leq 1 - (1 - \hat{p}_j(t))e_{lj} - p_l \qquad (10)$$
$$= 1 - (1 - \hat{p}_j(t))e_{lj} - (1 - (1 - p_j)e_{lj}) \qquad (11)$$
$$= (\hat{p}_j(t) - p_j)e_{lj} = e_{lj}\Delta_j(t) \leq e_{\max}\Delta^+(t), \qquad (12)$$

where (10) uses (3), (11) uses (9), (12) uses (1) and (4). ∎

Similarly, for $\Delta^-(t)$, we have the following lemma.

Lemma 2: Consider (3) under Assumption 1. With Δ^- defined in (4), if $\Delta^-(t) \neq 0$, then for all t,

$$\Delta^-(t+1) \leq e_{\max}\Delta^-(t). \qquad (13)$$

Proof: Consider $l \in \mathcal{K}_-(t+1)$. Suppose j is the constraining node of l at time $t + 1$, then

$$\Delta^-(t+1) \leq 1 - (1 - p_j)e_{lj} - \hat{p}_l(t+1) \qquad (14)$$
$$= 1 - (1 - p_j)e_{lj} - (1 - (1 - \hat{p}_j(t))e_{lj}) \qquad (15)$$
$$= (p_j - \hat{p}_j(t))e_{lj} = -e_{lj}\Delta_j(t) \leq e_{\max}\Delta^-(t), \qquad (16)$$

where (14) uses (2), (15) uses (3), (16) uses (1) and (4). ∎

Theorem 1 proves that (3) is regionally stable, in the sense that $\hat{p}_i(0) \in [0, 1)$, guarantees convergence to $\hat{p}_i = p_i$.

Theorem 1: Under conditions of Lemma 1 and Lemma 2,

$$L(t+1) \leq e_{\max}L(t), \;\; \forall t \geq t_0. \qquad (17)$$

Further $\hat{p}_i = p_i, \;\forall i \in V$ is the only stationary point of (3).

Proof: Lemmas 1 and 2 prove (17) when $L(t) > 0$. If $L(t) = 0$ then $\hat{p}_i(t) = p_i$, $\forall\ i \in V$ and from (3) and (2), for $i \in V$ and all t, $\hat{p}_i(t) = p_i$. Thus, (17) holds for $L(t) = 0$ as well and $\hat{p}_i \equiv p_i$ for all $i \in V$ is a stationary point of (3). Now suppose there exists another stationary point $\hat{p}_i = p_i^*$. Then $L(t) > 0$ and p_i^* cannot be a stationary point. ■

As $0 < e_{\max} < 1$ by (1), Theorem 1 establishes the regional exponential stability of (3). It is regional because we require $\hat{p}_i(0) \in [0, 1]$. Even though the Lyapunov function on [4] is the same, the result here is different. In [4], all that could be proved is that for some nonzero $a > 0$, $L(t + \mathcal{D}(G) - 1) \leq \max[0, L(t) - a]$. The decline proved is not at every instant. Further no bound on a is provided. In contrast here one has an exponential decline at a rate lower bounded by $e_{\max}$.

From lemmas 1 and 2 with $\Delta(t) = [\Delta_1(t), \ldots, \Delta_N(t)]^T$, $L_1(t) = \|\Delta(t)\|_\infty = \max\{\Delta^+(t), \Delta^-(t)\}$, also obeys $L_1(t+1) \leq e_{\max} L_1(t)$, i.e., $L_1(t)$ is an alternative Lyapunov function.

IV. Tight Bound on Time to Convergence

We now *tightly bound the convergence time*.

Theorem 2: Under Assumption 1, $\Delta^+(t)$ defined in (4) obeys

$$\Delta^+(t) = 0,\ \ \forall\ t \geq t_0 + \mathcal{D}(G) - 1. \tag{18}$$

Proof: As G is connected, each node belongs to a chain of nodes $n_1, n_2, \ldots, n_T$, such that n_i is a true constraining node of n_{i+1} and $n_1 \in S$. Note $T \leq \mathcal{D}(G)$. We prove

$$\Delta_{n_i}(t) \leq 0,\ \ \forall\ t \geq i - 1 + t_0,\ \text{ and } i \leq T. \tag{19}$$

Then the result is proved from (4). As $n_1 \in S$, (19) holds for $i = 1$. Now suppose it holds for some $i \in \{0, \ldots, T-1\}$. As $n_i \in \mathcal{C}(n_{i+1}) \subset \mathcal{N}(n_{i+1})$, from (3), (2) and the induction hypothesis that $\hat{p}_{n_i}(t) \leq p_{n_i}$ for $t \geq i + t_0$, for all $t \geq i+1+t_0$,

$$\begin{aligned}\hat{p}_{n_{i+1}}(t) &\leq 1 - (1 - \hat{p}_{n_i}(t-1)) e_{n_i n_{i+1}} \\ &\leq 1 - (1 - p_{n_i}) e_{n_i n_{i+1}} = p_{n_{i+1}}.\end{aligned}$$

Thus (19) holds and hence (18) is true. ■

Lemma 3 lower bounds the decay rate of $\Delta^-(t)$.

Lemma 3: Under the conditions of Lemma 2, suppose

$$\mathcal{S}^-(t) = \{i \in V | \Delta_i(t) < 0\} \neq \emptyset \tag{20}$$

and $\hat{p}_{\min}(t) = \min_{i \in \mathcal{S}^-(t)}\{\hat{p}_i(t)\}$. Then,

$$1 - \hat{p}_{\min}(t+1) \leq e_{\max}(1 - \hat{p}_{\min}(t)) \tag{21}$$

Proof: If $\mathcal{S}^-(t+1) \neq \emptyset$ is not empty then from Lemma 2 and (7), $\mathcal{S}^-(t) \neq \emptyset$. Suppose j is the constraining node of $i \in \mathcal{S}^-(t+1)$. Then we assert that $j \in \mathcal{S}^-(t)$. Indeed assume $j \notin \mathcal{S}^-(t)$. Thus $\hat{p}_j(t) \geq p_j$. As $j \in \mathcal{N}(i)$,

$$\hat{p}_i(t+1) = 1 - (1 - \hat{p}_j(t)) e_{ij} \geq 1 - (1 - p_j) e_{ij} \geq p_i.$$

Thus $i \notin \mathcal{S}^-(t+1)$, raising a contradiction. Thus,

$$\begin{aligned}1 - \hat{p}_i(t+1) = 1 - \hat{p}_{\min}(t+1) &= (1 - \hat{p}_j(t)) e_{ij} \\ &\leq (1 - \hat{p}_{\min}(t)) e_{ij} \leq (1 - \hat{p}_{\min}(t)) e_{\max}\end{aligned}$$

■

Thus, as $0 \leq \hat{p}_i(t) < 1$ for $i \in V$ and all t, $\hat{p}_{\min}(t)$ strictly increases unless $\mathcal{S}^-(t)$ is empty. We now tightly bound the time to convergence in the following theorem.

Theorem 3: Under conditions of Lemma 1 and Lemma 2, consider (3), $\mathcal{D}(G)$ defined in Definition 3 and $e_{\max}$ defined in (1). Define $p_{\max}(G) = \max_{i \in V}\{p_i\}$, for $G = (V, E)$. Then $L(t) = 0, \forall t \geq t_0 + T$ with T equaling

$$\max\left\{\mathcal{D}(G) - 1, T^- = \left\lceil \log_{e_{\max}}\left(\frac{1 - p_{\max}(G)}{1 - \hat{p}_{\min}(t_0)}\right)\right\rceil\right\}.$$

Further, for all $|V| > 3$, there exists a G satisfying Assumption 1 for which $L(t) > 0$ for all $t < T$.

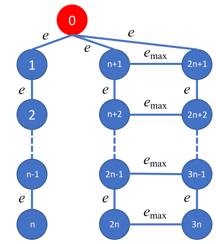

Fig. 1. Illustration of the tightness of convergence time. The entire graph is used in the proof of Theorem 3.

Proof: That $\Delta^+(t) = 0$ for all $t - t_0 \geq \mathcal{D}(G) - 1$ follows from Theorem 2. Suppose for some $t_1 \geq T^- + t_0$, $\Delta^-(t_1) > 0$. Then $1 - \hat{p}_{\min}(t_1) > 1 - p_{\max}(G)$. Then from Lemma 3, and the fact that $0 < e_{\max} < 1$, and dropping the arguments of $p_{\max}(G)$ and $\hat{p}_{\min}(t_0)$ in the last term,

$$\frac{1 - p_{\max}(G)}{1 - \hat{p}_{\min}(t_0)} < e_{\max}^{t_1 - t_0} \leq e_{\max}^{\log_{e_{\max}}\left(\frac{1 - p_{\max}(G)}{1 - \hat{p}_{\min}(t_0)}\right)} = \frac{1 - p_{\max}}{1 - \hat{p}_{\min}},$$

establishing a contradiction. Thus $\Delta^-(t) = 0$ for all $t \geq T^-$.

For tightness of T consider Figure 1 where $S = \{0\}$, $e_{\max} > e > 0$, $\hat{p}_i(t_0) > p_{\max}(G)\ \forall\ i \in \{1, \ldots, n\}$ and $\hat{p}_i(t_0) = 0,\ \forall\ i \in \{n+1, \ldots, 3n\}$. Observe, $\mathcal{D}(G) = n + 1$. As $p_0 = 0$, from (2)

$$(1 - p_k) = e^i,\ \forall\ k \in \{i, n+i, 2n+i\} \text{ and } i \in \{1, \ldots, n\}. \tag{22}$$

We now show by induction on t that for all $i \in \{1, \ldots, n\}$,

$$\hat{p}_i(t) > p_i\ \forall\ t \leq t_0 + i - 1. \tag{23}$$

If i's constraining node j, has $\hat{p}_j(t) > p_j$ then with $k \in \mathcal{C}(i)$

$$\hat{p}_i(t+1) > 1 - (1 - p_j)e \geq 1 - (1 - p_k)e = p_i. \tag{24}$$

Then (23) holds at $t = t_0$, as $\hat{p}_i(t_0) > p_{\max}(G)$ for all $i \in \{1, \ldots, n\}$. Suppose, it holds for some $\ell \in \{1, \ldots, n-1\}$ and all $t_0 \leq t \leq t_0 + \ell$. Consider $t_0 \leq t \leq t_0 + \ell + 1$. By the induction hypothesis, $\hat{p}_k(t_0 + \ell - 1) > p_k$ for all $k \in \mathcal{C}(i)$ and $i \in \{\ell + 1, \ldots, n\}$. Then from (24) $\hat{p}_i(t_0 + \ell) > p_i$ and $i \in \{\ell + 1, \ldots, n\}$ proving (23). We next show by induction on i that $\hat{p}_i(t) = p_i$ for all $i \in \{0, \ldots, n\}$ and $t \geq t_0 + i$. The result clearly holds for $i = 0$. Suppose it holds for some $\ell \in \{0, \ldots, n-1\}$ and $i \in \{0, \ldots, \ell\}$. From (22), $p_i > p_{i-1}$ for all $i \in \{0, \ldots, n\}$. Thus $\mathcal{C}(i) = \{i - 1\}$. Then because of (23) and the induction hypothesis, the result holds as for all $i \in \{0, \ldots, \ell + 1\}$ and $t \geq t_0 + \ell + 1$,

$$\hat{p}_i(t) = 1 - (1 - \hat{p}_{i-1}(t-1))e = 1 - (1 - p_{i-1})e = p_i.$$

Choose for some integer m, $e_{\max}^m = e$. Then the Appendix shows that for all $i \in \{1, \ldots, n\}$

$$\hat{p}_{n+i}(t) = \hat{p}_{2n+i}(t) = p_{n+i},\ \ \forall t \geq nm \tag{25}$$

and $\hat{p}_{2n}(t) < p_{2n}\ \forall t \leq nm$. From (22), $1 - p_{\max}(G) = e^n$. Then the result follows as T^- equals

$$\left\lceil \log_{e_{\max}}\left(\frac{1 - p_{\max}(G)}{1 - \hat{p}_{\min}(t_0)}\right)\right\rceil = \lceil \log_{e_{\max}} e_{\max}^{mn} \rceil = mn.$$

In Figure 1, $|V| = 3n+1$. For $|V| = 3n+2$ and $|V| = 3n+3$ add one and two sources respectively, that are connected only to 0. This does not change $\mathcal{D}(G)$ or $p_{\max}(G)$. ■

The proof shows that $\mathcal{D}(G) - 1$ is the tight bound on the convergence time of Δ^+ and T^- for Δ^-. Thus T is also a tight upper bound on the convergence of $L_1(t) = \|\Delta(t)\|_\infty$.

V. Robustness Under Perturbations

We now turn to the ultimate boundedness of the MPPA under *persistent perturbations on the* e_{ij} due to feedback or mobility. The changes from the nominal values e_{ij} are

$$\bar{e}_{ij}(t) = e_{ij} + \epsilon_{ij}(t). \tag{26}$$

Assume that with $e_{\min}$ and $e_{\max}$ in (1)

$$\exists\ \epsilon \text{ such that } |\epsilon_{ij}(t)| < \epsilon < \min\{e_{\min}, 1 - e_{\max}\}. \tag{27}$$

Thus $0 < \bar{e}_{ij}(t) < 1$. Bounded additive noise in communicating $\hat{p}_i$ to neighbors is equivalent to additive errors in e_{ij}. To accommodate this we permit asymmetric errors:

$$\bar{e}_{ij}(t) \neq \bar{e}_{ji}(t). \tag{28}$$

In particular, $\bar{e}_{ij}(t)$ is the value *seen by node* i *as opposed to node* j. Thus for all $i \notin S$, (3) is interpreted as:

$$\hat{p}_i(t+1) = \min_{j \in \mathcal{N}(i)} \{1 - (1 - \hat{p}_j(t))\bar{e}_{ij}(t)\}. \tag{29}$$

Definition 4: In G, call a path from a node i to the source an MPP, if it starts at i, ends in S and each node in the path is a true constraining node of its predecessor. We call an MPP from i, the longest such if it has the most number of nodes among all MPPs of i. Call $\mathcal{F}_k$ the set of nodes whose longest MPPs to the source have $k+1$ nodes.

If i has two most probable paths with two and three nodes then $i \notin \mathcal{F}_1$ but $i \in \mathcal{F}_2$. Further $\mathcal{F}_0 = S$ and $\forall i \in \{0, 1, \ldots, \mathcal{D}(G) - 1\}, \mathcal{F}_i \neq \emptyset$, and $\forall j \in \mathcal{F}_{k+1}, \mathcal{C}(j) \cap \mathcal{F}_k \neq \emptyset$. The smallest failure probability to a node in $\mathcal{F}_k$ is

$$p_{k\min} = \min_{j \in \mathcal{F}_k} \{p_j\}. \tag{30}$$

We make the following assumption.

Assumption 2: The graph G is connected and undirected; $S \neq V$; (26) under (27) and (1) holds. $e_{ij} = e_{ji}$. Further, $\forall i \in S, \hat{p}_i(t_0) = 0$ and $\forall i \in V \setminus S, \hat{p}_i(t_0) \in [0, 1)$. Without loss of generality, assume that $t_0 = 0$.

As $\bar{e}_{ij}(t) \in (0, 1)$, under Assumption 2, $\hat{p}_i(t)$ in (29) obeys

$$\hat{p}_i(t) \in [0, 1), \ \ \forall t \geq 0, \forall i \in V. \tag{31}$$

The following proves the ultimate boundedness of $\Delta^+(t)$.

Lemma 4: Suppose Assumption 2 holds. Consider (29). Then $\Delta^+(t) \leq \epsilon \sum_{i=0}^{\mathcal{D}(G)-2} (1 - p_{i\min})(e_{\max} - \epsilon)^{\mathcal{D}(G)-2-i} = L^+$ for all $t \geq \mathcal{D}(G) - 1$ with $\mathcal{D}(G)$ in Definition 3.

Proof: Consider the sequence of nodes $n_1, n_2, \ldots, n_T$ in the proof of Theorem 2. It follows that $T \leq \mathcal{D}(G)$. Then the result holds if for $t \geq i-1$ and $i \in \{1, \ldots, T\}$

$$\hat{p}_{n_i}(t) \leq p_{n_i} + \epsilon \sum_{l=0}^{i-2} (1 - p_{l\min})(e_{\max} - \epsilon)^{i-2-l}. \tag{32}$$

We prove (32) by induction. It holds for $i = 1$ by (2) and (29). Suppose (32) holds for some $i \in \{1, \ldots, T-1\}$. From (29) and our induction hypothesis, for all $t \geq i$,

$$\begin{aligned}
\hat{p}_{n_{i+1}}(t) &\leq 1 - \big(1 - \hat{p}_{n_i}(t-1)\big)\bar{e}_{n_{i+1}n_i}(t-1) \\
&\leq 1 - \big(1 - \hat{p}_{n_i}(t-1)\big)(e_{n_{i+1}n_i} - \epsilon) \qquad (33)\\
&\leq 1 - \left(1 - p_{n_i} - \epsilon \sum_{l=0}^{i-2}(1 - p_{l\min})(e_{\max} - \epsilon)^{i-2-l}\right) \\
&\quad \times (e_{n_{i+1}n_i} - \epsilon) \qquad (34)\\
&= p_{n_{i+1}} + (1 - p_{n_i})\epsilon \\
&\quad + (e_{n_{i+1}n_i} - \epsilon)\epsilon \sum_{l=0}^{i-2}(1 - p_{l\min})(e_{\max} - \epsilon)^{i-2-l} \\
&\leq p_{n_{i+1}} + (1 - p_{i-1\min})\epsilon \\
&\quad + (e_{\max} - \epsilon)\epsilon \sum_{l=0}^{i-2}(1 - p_{l\min})(e_{\max} - \epsilon)^{i-2-l} \\
&= p_{n_{i+1}} + \epsilon \sum_{l=0}^{i-1}(1 - p_{l\min})(e_{\max} - \epsilon)^{i-1-l} \qquad (35)
\end{aligned}$$

where (33) uses (26), (27) and the fact that $0 \leq \hat{p}_{n_i}(t-1) < 1$ by (31), (34) uses $e_{n_{i+1}n_i} > \epsilon$ and (35) uses (2). ■

For $\Delta^-(t)$, we use a variant of the *comparison principle*, by using an *expanded graph*, $G^+ = (V, E^+)$ defined below.

Definition 5: Given $G = (V, E)$, the undirected graph $G^+ = (V, E^+)$, has the property that edge $(i, j) \in E^+$ iff $(i, j) \in E$. The edge value e_{ij}^+ between nodes i and j obeys $e_{ij}^+ = e_{ij} + \epsilon$, with ϵ defined in (27). Further G^+ has the same source set S as G, each i has the same set of neighbors as in G, and P_i is the probability of failure from the source set to i, i.e., plays the role of p_i in G. We define $\mathcal{D}(G^+)$ as the effective diameter of G^+, and $\mathcal{F}_i^+$ the set of nodes whose longest MPPs have $i+1$ nodes in G^+.

We further define

$$p_{i\min}^+ = \min_{j \in \mathcal{F}_i^+} \{P_j\} \in [0, 1), \tag{36}$$

Note $p_{0\min}^+ = 0$ as $\mathcal{F}_0^+ = S$. Define $\hat{P}_i(t)$ as the estimated failure probability of $i \in V$ in G^+. As G^+ is fixed, we obtain

$$\hat{P}_i(t+1) = \begin{cases} \min\limits_{j \in \mathcal{N}(i)} \{1 - (1 - \hat{P}_j(t)) \cdot (e_{ij} + \epsilon)\} & i \notin S \\ 0 & i \in S \end{cases} \tag{37}$$

Lemma 5 establishes a relation between in G and G^+.

Lemma 5: Under Assumption 2, consider $\hat{P}_i(t)$ defined in (37) and $\hat{p}_i(t)$ defined in (29). Suppose for all $i \in V$, $\hat{p}_i(0) \geq \hat{P}_i(0)$. Then $\hat{p}_i(t) \geq \hat{P}_i(t)$ for all $i \in V$ and $t \geq 0$.

Proof: Use induction. The result holds for $t = 0$. Suppose for some $t \geq 0$, $\hat{p}_i(t) \geq \hat{P}_i(t)$ for all $i \in V$. Suppose $j \in \mathcal{N}(i)$ is a constraining node of i at time $t+1$ in (29). Then:

$$\begin{aligned}
\hat{P}_i(t+1) &\leq 1 - (1 - \hat{P}_j(t))(e_{ij} + \epsilon) \qquad (38)\\
&\leq 1 - (1 - \hat{p}_j(t))(e_{ij} + \epsilon) \qquad (39)\\
&\leq 1 - (1 - \hat{p}_j(t))\bar{e}_{ij}(t) = \hat{p}_i(t+1) \qquad (40)
\end{aligned}$$

where (38) uses (37), (39) uses the induction hypothesis, and (40) uses (27) and the fact that $0 \leq \hat{p}_j(t) < 1$. ■

Thus, the probability estimates $\hat{p}_i(t)$ in (29) are lower bounded by $\hat{P}_i(t)$ in (37) for $\hat{p}_i(0) \geq \hat{P}_i(0)$. Further, as G^+ satisfies the same assumptions as G, and is perturbation free, both $\Delta^+(t)$ and $\Delta^-(t)$ of (37) converge to zero. Lemma 6 obtains ultimate bound on Δ^- of (29), by relating P_i to p_i.

Lemma 6: Suppose Assumption 2 holds. Then for all $i \in V$,

$$p_i \le P_i + \epsilon \sum_{l=0}^{\mathcal{D}(G^+)-2} (1 - p^+_{l\,\min}) e_{\max}^{\mathcal{D}(G^+)-2-l}. \quad (41)$$

Proof: Consider $n_1, n_2, \dots, n_{T_1}$, such that $n_1 \in S$, and for all $i \in \{1, \dots, T_1 - 1\}$, n_i is a true constraining node of n_{i+1} in G^+. Every node in G^+ is in one such sequence, and $T_1 \le \mathcal{D}(G^+)$. Then the result holds if for all $i \in \{1, \dots, T_1\}$,

$$p_{n_i} \le P_{n_i} + \epsilon \sum_{l=0}^{i-2} (1 - p^+_{l\,\min}) e_{\max}^{i-2-l}, \quad (42)$$

where the summation is zero if the lower limit exceeds the upper. As $p_{n_1} = P_{n_1} = 0$, (42) holds for $i = 1$. Suppose (42) holds for some $i \in \{1, \dots, T_1 - 1\}$. As n_i and n_{i+1} are neighbors in both G and G^+, n_i is a true constraining node of n_{i+1} in G^+, (27) and (36) hold, $p_{n_{i+1}}$ obeys

$$\begin{aligned}
p_{n_{i+1}} &\le 1 - (1 - p_{n_i}) e_{n_{i+1} n_i} \\
&\le 1 - \Big(1 - P_{n_i} - \epsilon \sum_{l=0}^{i-2} (1 - p^+_{l\,\min}) e_{\max}^{i-2-l}\Big) e_{n_{i+1} n_i} \\
&= 1 - (1 - P_{n_i})(e_{n_{i+1} n_i} + \epsilon) + (1 - P_{n_i})\epsilon \\
&\quad + e_{n_{i+1} n_i} \epsilon \sum_{l=0}^{i-2} (1 - p^+_{l\,\min}) e_{\max}^{i-2-l} \\
&\le P_{n_{i+1}} + (1 - p^+_{i-1\,\min})\epsilon + e_{\max} \epsilon \sum_{l=0}^{i-2} (1 - p^+_{l\,\min}) e_{\max}^{i-2-l} \\
&= P_{n_{i+1}} + \epsilon \sum_{l=0}^{i-1} (1 - p^+_{l\,\min}) e_{\max}^{i-1-l}
\end{aligned}$$

Thus (42) holds and our result follows. ■

We now prove the ultimate boundedness of Δ^-.

Lemma 7: Consider (29) under Assumption 2, with $p_{\max}(G^+)$ defined in Theorem 3. Then for all

$$t \ge T_1 = \left\lceil \log_{e_{\max}+\epsilon} \left(\frac{1 - p_{\max}(G^+)}{1 - \hat{p}_{\min}(0)} \right) \right\rceil, \quad (43)$$

there holds:

$$\Delta^-(t) \le \epsilon \sum_{l=0}^{\mathcal{D}(G^+)-2} (1 - p^+_{l\,\min}) e_{\max}^{\mathcal{D}(G^+)-2-l} = L^-. \quad (44)$$

Proof: Consider (37) in G^+ with the initialization in Lemma 5. As G^+ satisfies Assumption 1, $\hat{P}_i(t) \ge P_i$ for all $i \in V$ after a finite time T_1. Thus from Lemma 5, $\hat{p}_i(t) \ge P_i$ for all $t \ge T_1$ and $i \in V$. From Lemma 6, for $t \ge T_1$,

$$\begin{aligned}
-\Delta_i(t) &\le p_i - P_i \\
&\le P_i + \epsilon \sum_{l=0}^{\mathcal{D}(G^+)-2} (1 - p^+_{l\,\min}) e_{\max}^{i-2-l} - P_i \\
&= \epsilon \sum_{l=0}^{\mathcal{D}(G^+)-2} (1 - p^+_{l\,\min}) e_{\max}^{i-2-l}.
\end{aligned}$$

From Theorem 3, T_1 is the time when $\Delta^-(t)$ of (37) goes to zero. With Lemma 3 and Theorem 3, T_1 obeys (43). ■

Now the tightness of the bound and the time to attain it:

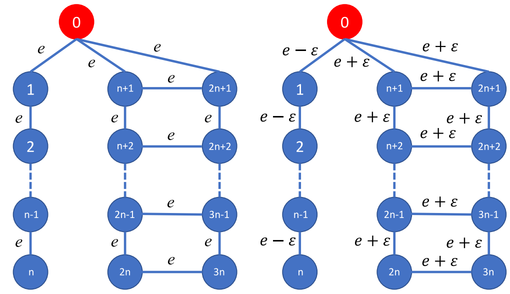

Fig. 2. Illustration of the tightness of the ultimate bounds and the time to attain the ultimate bounds under perturbations.

Theorem 4: Consider (29), the conditions of Lemma 4 and Lemma 7. Then with L^+ and L^- defined in lemmas 4 and 7 respectively, $L(t)$ in (5) obeys $L(t) \le L^+ + L^-$, $\forall t \ge T'$. With T_1 defined in (43), $T' = \max\{\mathcal{D}(G) - 1, T_1\}$. For every $|V| > 3$, there exist G and perturbations obeying Assumption 2 for which the bound is attained in precisely T' steps.

Proof: Ultimate boundedness follows from Lemma 4 and 7. For the tightness of the ultimate bound and the time to attain it, consider the graphs in Figure 2, the right the perturbed version of the left. Assumptions 2 and 1 hold. The probability estimates in the perturbed graph converge to their correct values as the e_{ij} are fixed. As $e = e_{\max}$ in this case, substituting e by $e_{\max} - \epsilon$ in (22) we get $p_n = 1 - (e_{\max} - \epsilon)^n$ and $\Delta^+ = e_{\max}^n - (e_{\max} - \epsilon)^n = \epsilon \sum_{i=0}^{n-1} e_{\max}^i (e_{\max} - \epsilon)^{n-1-i}$. Further as $p_{i\,\min} = 1 - e_{\max}^n$ and $\mathcal{D}(G) = n+1$ this equals L^+ in Lemma 4. From the proof of Theorem 3 this is attained in exactly $\mathcal{D}(G) - 1$ steps provided $\hat{p}_i(0) > p_n$ for all $i \in \{1, \dots, n\}$. Substituting e by $e_{\max} + \epsilon$ in (22), $p_{2n} = p_{3n} = 1 - (e_{\max} + \epsilon)^n$. It is readily checked that $2n$ has the greatest underestimate. Using the fact that $p^+_{l\min} = 1 - (e_{max} + \epsilon)^l$ a similar argument as above shows that the L^- is precisely met and as in the proof of Theorem 3 in exactly T_1 steps provided $\hat{p}_i(0) = 0$ for all $i \in \{n+1, \dots, 3n\}$. For $|V| \ne 3n + 1$ use same devices as in the proof of Theorem 3. ■

This establishes a tight ultimate bound and a tight bound on the time to achieve it. Again, from the proof of this theorem, $\max\{L^+, L^-\}$ is a tight ultimate bound for $L_1(t) = \|\Delta(t)\|_\infty$, while the tight bound on its time to attainment is still T'. We can also obtain MPPs using ABF by minimizing $-\log(1 - p_i) = -\log q_i$. In this case $\hat{q}_i(t)$, the estimate of q_i, which is 0 if $i \in S$ and $\min_{j \in \mathcal{N}(i)} \{-\log \hat{q}_j(t) - \log e_{ij}\}$ otherwise, is identical to ABF with same convergence time as for (3). However, as the perturbed update is $\log[e_{ij} + \epsilon_{ij}] - \log e_{ij}$, [4, Lemma 5] and consequently the tight ultimate bounds obtained for ABF in [4], do not hold.

VI. Simulations

We use a graph with 500 nodes two being sources, uniformly distributed in a 4×1 km^2 area, communicating in a 0.25 km radius. We run 5 times with asymmetric noise in the estimated e_{ij} which change from their nominal values as in (26), and the noise $\epsilon_{ij}(t)$ is $\mathcal{U}[-\min\{e_{\min}, 1 - e_{\max}\}, \min\{e_{\min}, 1 - e_{\max}\}]$, thus obeying (27).

Simulation results are shown in Figure 3, $\epsilon = 5.8 \times 10^{-5}, 2.7 \times 10^{-4}, 7.0 \times 10^{-5}, 1.1 \times 10^{-4}$ and 2.3×10^{-5} for those 5 trials. $\Delta^+(t)$ becomes lower than its predicted ultimate bound after 20 rounds with $\mathcal{D}(G)$ ranging from 20 to

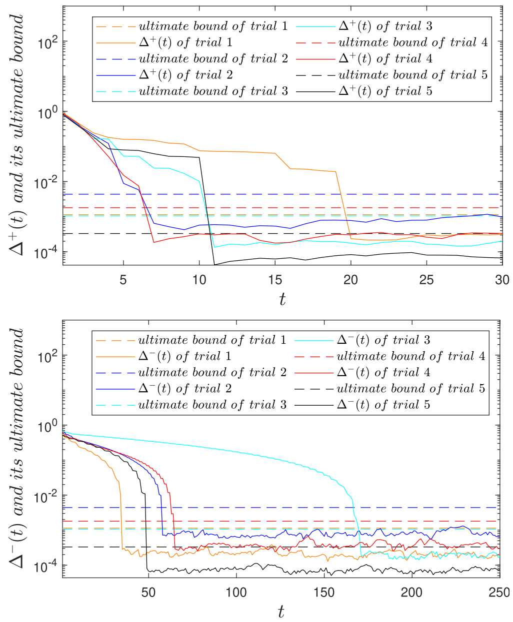

Fig. 3. Trace of (a) $\Delta^+(t)$ and its ultimate bound, and (b) $\Delta^-(t)$ and its ultimate bound for 5 runs.

27: $\Delta^-(t)$ takes longer to drop below its ultimate bound. This is so as from (43) in Lemma 7: The time for $\Delta^-(t)$ to be ultimately bounded depends on factors beyond $\mathcal{D}(G)$.

VII. Conclusion

We have proved the regional stability of MPPA by using a Lyapunov approach. We have shown that the algorithm is ultimately bounded under persistent perturbations, have provided tight bounds on the ultimate bounds and the time to attain them in terms of global parameters like $\mathcal{D}(G)$.

Though Bellman's equations like (2), though not (2) itself, have been analyzed, barring [4], Lyapunov analysis is rare. Lyapunov analyses permit the establishment of tight ultimate bounds under persistent perturbations. As described in [4], while there is perturbation analysis of Bellman's equations, e.g., in path finding, such analyses assume that perturbations in the shortest path from a source to a destination have subsided.

Further, aggregate computing requires the analysis of block compositions of multiple Bellman's equations, frequently under feedback. A Lyapunov framework together and tight ultimate bounds open up the prospect *non-conservative* close loop analysis using Lyapunov-based small gain theorems, [5].

Appendix

Proof of (25) and the fact that $\hat{p}_{2n}(t) < p_{2n}$, $\forall\ t < nm$: By symmetry $\hat{p}_{n+i}(t) = \hat{p}_{2n+i}(t)$. Call $n+i$ and $2n+i$ partners. From (22) the result holds if for all $i \in \{1, \ldots, n\}$,

$$\hat{p}_{n+i}(t) = 1 - e^t,\ \forall t < mi,\ \hat{p}_{n+i}(t) = p_{n+i}, \forall t \geq mi. \quad (45)$$

From (3), partners constrain each other until for some $j \in \mathcal{N}(n+i)$, $(1-\hat{p}_{n+i}(t))e_{\max} < (1-\hat{p}_j(t))e$, i.e.,

$$1 - \hat{p}_{n+i}(t) < (1-\hat{p}_j(t))e_{\max}^{m-1},\ \text{for some } j \in \mathcal{N}(i). \quad (46)$$

As $\hat{p}_0(0) = \hat{p}_{n+i}(0) = \hat{p}_{2n+i}(0)$, (46) is violated at $t = 0$ and from (3), $1 - \hat{p}_{n+i}(t) = e_{\max}^t$, until the partners stop constraining each other. We prove (45) using induction on i.

Note $2n+i$ is a valid constraining node of $n+i$ as long as all of $n+i$'s neighbors have estimates $1-e^t$. Only $n+1$ and $2n+1$ have edges to a node with a different estimate, namely 0. From (46), $n+1$ and $2n+1$ stop constraining each other when $e^t > e_{\max}^{m-1}$, i.e., at $t = m$. In fact 0 constrains $n+1$ and $2n+1$ for all $t \geq m$. Thus as $\hat{p}_0(t) = 0$, $\hat{p}_{n+1}(t) = 1-e$ and from (22), (45) holds for $i = 1$, initiating the induction.

Suppose the result holds for some $\ell \in \{2, \ldots, n\}$ and all $i \in \{1, \ldots, \ell-1\}$. By the induction hypothesis $1-\hat{p}_{n+\ell-1}(t) = e_{\max}^{m(\ell-1)}$, $\forall\ t \geq m(\ell-1)$. For all $j \geq \ell$, $2n+j$ constrains $n+j$ until $n+\ell$ is constrained by $n+\ell-1$. From (46) this will happen if $e_{\max}^t < e_{\max}^{m(\ell-1)+m-1}$, i.e., at $t = m\ell$. At this point while the nodes $n+j$, $j > \ell$ are still constrained by their partner, (45) holds for $i = \ell$ as from (3), $\hat{p}_{n+\ell}(t) = 1 - e^{\ell-1}e = p_\ell$, $\forall t \geq m\ell$.

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References

[1] M. Viroli, G. Audrito, J. Beal, F. Damiani, and D. Pianini, "Engineering resilient collective adaptive systems by self-stabilisation," *ACM Trans. Model. Comput. Simulat.*, vol. 28, no. 2, pp. 1–28, 2018.
[2] A. Zanella, N. Bui, A. Castellani, L. Vangelista, and M. Zorzi, "Internet of Things for smart cities," *IEEE IoT J.*, vol. 1, no. 1, pp. 22–32, Feb. 2014.
[3] Y. Mo, S. Dasgupta, and J. Beal, "Robust stability of spreading blocks in aggregate computing," in *Proc. IEEE Conf. Decis. Control (CDC)*, 2018, pp. 6007–6012.
[4] Y. Mo, S. Dasgupta, and J. Beal, "Robustness of the adaptive Bellman-Ford algorithm: Global stability and ultimate bounds," *IEEE Trans. Autom. Control*, vol. 64, no. 10, pp. 4121–4136, Oct. 2019.
[5] Z.-P. Jiang, I. M. Mareels, and Y. Wang, "A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems," *Automatica*, vol. 32, no. 8, pp. 1211–1215, 1996.
[6] E. W. Dijkstra, "A note on two problems in connexion with graphs," *Numerische Mathematik*, vol. 1, no. 1, pp. 269–271, 1959.
[7] R. Bellman, "On a routing problem," *Quart. Appl. Math.*, vol. 16, no. 1, pp. 87–90, 1958.
[8] H. Khalife, S. Ahuja, N. Malouch, and M. Krunz, "Probabilistic path selection in opportunistic cognitive radio networks," in *Proc. IEEE Global Telecommun. Conf.*, 2008, pp. 4861–4865.
[9] D. Brockmann and D. Helbing, "The hidden geometry of complex, network-driven contagion phenomena," *Science*, vol. 342, no. 6164, pp. 1337–1342, 2013.
[10] S. Ekisheva and M. Borodovsky, "Probabilistic models for biological sequences: selection and maximum likelihood estimation," *Int. J. Bioinform. Res. Appl.*, vol. 2, no. 3, pp. 305–334, 2006.
[11] M. Albared, N. Omar, M. J. Ab Aziz, and M. Z. A. Nazri, "Automatic part of speech tagging for arabic: An experiment using bigram hidden Markov model," in *Proc. Int. Conf. Rough Sets Knowl. Technol.*, 2010, pp. 361–370.
[12] E. Ser-Giacomi, R. Vasile, E. Hernández-García, and C. López, "Most probable paths in temporal weighted networks: An application to ocean transport," *Phys. Rev. E, Stat. Phys. Plasmas Fluids Relat. Interdiscip. Top.*, vol. 92, no. 1, 2015, Art. no. 012818.